Key

Final Exam

Math 200
(Fall 2005)

Solve the following problems. Show all your work in the space under each problem.

1. Use partial fractions to find the sum of the series $\sum_{k=3}^{\infty} \frac{1}{k^2 - k}$. (2 pts)

$a_k = \frac{1}{k^2-k} = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

$S_n = a_3 + a_4 + a_5 + \cdots + a_n = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n-1} - \frac{1}{n}$
$= \frac{1}{2} - \frac{1}{n}$

$\sum_{k=3}^{\infty} \frac{1}{k^2-k} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{n}\right] = \boxed{\frac{1}{2}}$

2. True or False: If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Is the converse true? If your answer is "No" provide a counterexample. (3 pts)

$\boxed{\text{True}}$.

$\boxed{\text{No}}$, eg $\sum_{k=0}^{\infty} \frac{1}{k}$. Then, $\lim_{k\to\infty} \frac{1}{k} = 0$, but $\sum_{k=0}^{\infty} \frac{1}{k}$ diverges

3. Use the Integral Test to determine whether the series $\sum_{k=1}^{\infty} ke^{-k^2}$ converges or diverges. (2 pts)

(*Hint*: For the integral, you don't need to use by parts. Just think what the derivative of e^{-x^2} is)

$\int_1^{\infty} xe^{-x^2}dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{2}\int_1^b -2xe^{-x^2}dx\right]$
$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b$
$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}\right]$
$= \frac{1}{2}e^{-1}$

ie, $\int_1^{\infty} xe^{-x^2}dx$ converges.

Hence, $\sum_{k=1}^{\infty} ke^{-k^2}$ converges.

4. True or False: If $\sum_{k=0}^{\infty} |a_k|$ converges, then $\sum_{k=0}^{\infty} a_k$ converges. Is the converse true? If your answer is "No" provide a counterexample. (3 pts)

True

No, e.g. $\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1}$. Then, $\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1}$ converges, but $\sum_{k=0}^{\infty}\left|(-1)^k\frac{1}{k+1}\right| = \sum_{k=0}^{\infty}\frac{1}{k+1}$ diverges.

5. Find the Taylor series of $f(x)=e^{2x}$ at $x_0 = 1$. Make sure you include the n^{th}-term of the series. (2 pts)

$f(x) = e^{2x} \Rightarrow f(1) = e^2$

$f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$

$f''(x) = 2.2e^{2x} \Rightarrow f''(1) = \underbrace{2.2}_{2^2}e^2$

$f'''(x) = 2.2.2e^{2x} \Rightarrow f'''(1) = \underbrace{2.2.2}_{2^3}e^2$

$\vdots$

$f^{(n)}(x) = \underbrace{2.2 \cdots 2}_{n}e^{2x} \Rightarrow f^{(n)}(1) = 2^n e^2$

$\Rightarrow$ So, $f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \cdots$

$= e^2 + \frac{2e^2}{1!}(x-1) + \frac{2^2e^2}{2!}(x-1)^2 + \frac{2^3}{3!}e^2(x-1)^3 + \cdots + \frac{2^ne^2}{n!}(x-1)^n + \cdots$

6. Evaluate the following double integrals: (4 pts)

(a) $\iint_R e^{x+y}\,dydx$, $R: 0 \le x \le 1, 0 \le y \le 2$ (b) $\iint_R x^2\,dxdy$, $R: 0 \le x \le 1, 0 \le y \le 3$

(a) $\int_0^1\int_0^2 e^{x+y}\,dydx = \int_0^1 \left[e^{x+y}\right]_0^2 dx$

$= \int_0^1 (e^{x+2} - e^x)\,dx$

$= \left[e^{x+2} - e^x\right]_0^1$

$= \boxed{e^3 - e - e^2 + 1}$

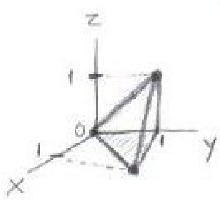
(b) $\int_0^3\int_0^1 x^2\,dxdy = \int_0^3 \left[\frac{x^3}{3}\right]_0^1 dy$

$= \int_0^3 \left(\frac{1}{3}\right) dy$

$= \left[\frac{1}{3}x\right]_0^3$

$= \boxed{1}$

7. Use triple integration to find the volume of the tetrahedron D with vertices (0,0,0), (1,1,0), (0,1,0) and (0,1,1). (4 pts)
(*Hint*: The upper bounding surface is the plane $z = y - x$ and the lower bounding surface is the xy-plane)



$V = \int_0^1 \int_{y=x}^{y=1} \int_{z=0}^{z=y-x} 1\,dz\,dy\,dx$

$= \int_0^1 \int_{y=x}^{y=1} \left[z\right]_{z=0}^{z=y-x} dydx$

$= \int_0^1 \int_{y=x}^{y=1} (y-x)\,dy\,dx$

$= \int_0^1 \left[\frac{y^2}{2} - xy\right]_{y=x}^{y=1} dx = \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right)dx = \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1$

$= \boxed{1/6}$